

Univalent Non-Associative Categorical Structures

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Abstract. We study category-like structures in which composition is not associative, especially Munch-Maccagnoni’s duploids, in univalent foundations. This includes a definition of what it means for a duploid to be *univalent*: that isomorphisms of a suitable kind render the objects indistinguishable to the type theory. We show that the existing construction of a duploid from an adjunction fails to be univalent as a "single-sorted" duploid, and provide a novel construction which is. This work is formalized in the `UNIMATH` library for the `Rocq` proof assistant.

Keywords: Univalence · Duploids · Adjunctions · Polarization

1 Introduction

In this paper we study two non-associative categorical structures: *unital magmoids* and *duploids*. A unital magmoid consists of the same data as a category, but does not require composition to be associative. A duploid is special kind of unital magmoid in which objects are "polarized", and which is equipped with an extra "polarity shift" structure [16]. Polarization in this definition corresponds to the recovery of associativity for certain composites, as developed in Munch-Maccagnoni’s PhD thesis [15], but has many guises.

In the verbiage of call-by-push-value [8], *computations* are negative, and *values* are positive. Polarity shifts axiomatise *thunks* – values which are frozen computations – and their dual, *returners* – computations which return values.

Duploids have been used to study and unify the theory of polarization in programming languages, logic, and their categorical models [15,5,12,11]. As a pertinent example, where one may construct a Kleisli category from a monad to model call-by-value [14,16,6], and a co-Kleisli category from a comonad to model call-by-name [12,15], one may construct a duploid from an adjunction which contains both the Kleisli and co-Kleisli categories as subcategories.

Univalence In set-theory, *equality* (or *identity*, written $=$) is a global equivalence relation which renders the related objects indistinguishable to the logic. In univalent foundations, the notion of identity, while still rendering the objects indistinguishable, is less strict than equality in set-theoretic foundations. In exchange, identity in univalent foundations is able to coincide with weaker notions of equivalence, such as *isomorphism* (of sets, groups, rings, *etc.*) or *adjoint equivalence* (of categories).

As an example, it is well-known that in a category $\mathcal{C}$, the correct notion of equivalence between two objects $a, b : \mathcal{C}$ is *isomorphism* $a \cong b$, and not necessarily identification $a = b$. In set-theoretic foundations, equality is typically overly restrictive: there may be

objects $a \cong b$ that are not equal,¹ and there may be isomorphisms $a \cong a$ that are not the identity.² In univalent foundations, the type of identifications $a =_{\text{ob } C} b$ may very well correspond to isomorphisms $a \cong b$. A *univalent category* is one where they do [2]. As an example, a partial order is a univalent category, even in set-theoretic foundations.

Contribution We study non-associative composition from a univalent point of view: we define what it means for unital magmoids and duploids to be *univalent*, and take care to construct things to satisfy and benefit from univalence. We must already be careful, for there is a bifurcation of definitions of "duploid", between "single-sorted" and "two-sorted" variants, and they accordingly have different definitions of univalence.

Univalence is defined based on a well-behaved subset of isomorphisms. We find that such isomorphisms in a unital magmoid must be not only linear [16] and thunkable [6], but also satisfy a third property which we call *intermediate*. In a duploid, intermediateness is trivialised for isomorphisms that are both linear and thunkable already.

We point out that the original construction of a duploid $\mathcal{D}$ from an adjunction introduces extra structure on objects that is not invariant under (linear-and-thunkable) isomorphism. The issue manifests as non-univalence of $\mathcal{D}$ when viewed as a unital magmoid. Our main result is the construction of an alternative duploid from an adjunction, which does satisfy our definition of univalence for unital magmoids.

Our work is formalized in the Rocq library `Unimath`. The formalization diverges from trunk commit `756380729085` and constitutes a change of 13591 lines of code added across 36 files, as counted by `CLOC` [4]. The material of this work is located in the directory `Unimath/CategoryTheory/Nonassociative/`. The proofs and definitions presented in this text are based on the formalization, but may differ, or be presented differently, for clarity.

Related Work This paper is primarily based on Munch-Maccagnoni's *Models of a Non-Associative Composition* [16,15], which introduces duploids as a mathematical structure. Much of our work is based on formalizing the definitions and proofs of *op. cit.* in Rocq. The parts that we do not include are the internal language of a duploid, and the full *structure theorem* relating the category of duploids to the category of adjunctions.

The rest of Munch-Maccagnoni's PhD thesis [15] explores non-associative composition and polarization more broadly, relating it to programs, with first-class continuations/continuation-passing-style translations, and to proofs, with the involutive negation in classical logic. Other work on duploids includes that of Mangel *et al.* [12] about linear classical logic, which also includes some general theory about duploids. We note that our "correct notion of isomorphism" in a unital magmoid (Definition 6) differs from theirs, but they do coincide for preduploids (Corollary 2).

Duploids are related to Levy's call-by-push-value and its adjunction models [8,9]. Especially pertinent are Levy's *Contextual Isomorphisms* [10]. Levy defines *contextual isomorphism* of types $\theta : A \cong B$ syntactically with respect to an equivalence relation $\sim$ on typed terms. Given a typing judgement with a type hole $\Gamma[-] \vdash C[-]$, a contextual

¹ Say, the sets $\{0, 1\}$ and $\{1, 2\}$.

² Say, the set $\{\perp, \top\}$ under boolean negation.

isomorphism bijectively maps $\sim$ -classes of typed terms $\Gamma[A] \vdash t : C[A]$ to $\sim$ -classes of typed terms $\Gamma[B] \vdash \theta(t) : C[B]$. Levy shows for call-by-push-value that when $\sim$ includes computation rules, contextual isomorphisms correspond to thunkable (Führmann [6]) isomorphisms for value (positive) types, and linear (Munch-Maccagnoni [16]) isomorphisms for computation (negative) types. This result is similar to our definition of univalence for duploids in subsection 4.2, and our proofs of univalence in subsection 5.4.

For the "univalent" part, this paper is based on the study of category theory in univalent foundations initiated by Ahrens *et al.* [2]. Our work is also based on *op. cit.* in the literal sense that we directly use its Rocq formalization, which is today part of UNIMATH [7]. Areas of category theory already studied in univalent foundations include bicategories [1], monoidal categories [20] and monads [19].

Another work is *The Univalence Principle* by Ahrens *et al.* [3], deriving algorithmically what it means for a large class of mathematical structures to be "univalent". Ahrens *et al.* mention duploids, but only the "two-sorted" variant (see Remark 3). Still, our definitions of "univalent duploid" (Definitions 12 and 14) are the same as derived by their procedure [3, Example 13.5].

2 Preliminary Type Theory

Our theorems, definitions, and proofs are written, and to be understood, as informal mathematical prose. We do adopt the vernacular, and some notation, of homotopy type theory [18, 17], which we will detail here. Set theorists will miss out on only some of this paper by reading "*is a proposition*" as "is unique if it exists", "*equivalence (of types)*" as "isomorphism (of sets)", and "*is a set*" as tautological.³

As in the UNIMATH [7] formalization, we work in Martin-Löf type theory [13] with intensional identity types and no W-types. Unlike the UNIMATH formalization, we do not assume any universes unless stated. We freely assume function extensionality.

We write products $\prod_{x:B} E[x]$, sums $\sum_{x:B} E[x]$, binary products $A \times B$, binary sums $A \amalg B$, non-dependent function types $A \rightarrow B$, and finite types as sets $\{a, b, \dots\}$. Judgemental equality is written with $\equiv$ and the identification type with $=_A$, or simply $=$.

A type is *contractible* if it has exactly one element, and a *proposition* if it has at most one element. A type A is a *set* if for all $a, b : A$ the identification type $a =_A b$ is a proposition. The *propositional truncation* of a type is the proposition that it is inhabited. The inhabitant of a type may be obtained from the propositional truncation so long as the representative is irrelevant: a principle called *unique choice*.

Convention 1. The phrase "there exists a unique" is to be read as saying the subsequent type is contractible. Unlike the *HoTT Book* [18], we use propositionally truncated interpretations of logic: "there exists" is to be read as propositional truncation; "or" is a binary sum, and under propositional truncation unless preceded by "either".

Definition 1 ([18]). A function $f : A \rightarrow B$ is an *equivalence (of types)*, written $f : A \simeq B$, if for all $y : B$ there exists a unique $x : A$ with an identification $f(x) = y$. Every equivalence $f : A \simeq B$ has a two-sided inverse $f^{-1} : B \rightarrow A$, and every function with a two-sided inverse may be upgraded to an equivalence.

³ Our results are in fact consistent, though less interesting, with uniqueness of identity proofs.

3 Unital Magmoids

In this section we recall the definition of a univalent category, and generalize it to unital magmoids. Along the way, we introduce Munch-Maccagnoni's characterization of polarity as the failure of associativity [16,15], as it is necessary to define the correct notion of isomorphism in a unital magmoid.

A unital magmoid consists of the same data as a category [2], but need not satisfy associativity of composition. Explicitly:

Definition 2. A unital magmoid or non-associative category $\mathcal{M}$ consists of the following:

1. A type of objects, written $\mathbf{ob} \mathcal{M}$ or just $\mathcal{M}$.
2. For all $a, b : \mathbf{ob} \mathcal{M}$, a type of morphisms from a to b , also known as a hom-type, written $\mathcal{M}[[a, b]]$ or $a \rightarrow b$.
3. For all $a : \mathbf{ob} \mathcal{M}$, an identity morphism $\text{id}_a : a \rightarrow a$.
4. A composition operator $\circ$ assigning to each pair of morphisms $f : a \rightarrow b$ and $g : b \rightarrow c$, a composite $f \circ g : a \rightarrow c$.
5. Composition must be unital: $\text{id}_a \circ f = f$ and $f \circ \text{id}_b = f$ for all $f : a \rightarrow b$.
6. $\mathcal{M}$ must have hom-sets: each hom-type $\mathcal{M}[[a, b]]$ is a set.⁴

We shall import the notion of "isomorphism" $a \cong b$ for unital magmoids from categories directly, even though they are less well-behaved. For example, inverses are not necessarily unique in a unital magmoid. To obtain a better-behaved notion, we must require certain composites to associate.

Definition 3 ([16]). A path of three morphisms $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ in a unital magmoid is said to associate when $f \circ (g \circ h) = (f \circ g) \circ h$. We say that:

- f is thunkable if all triples (f, g, h) associate.
- g is intermediate if all triples (f, g, h) associate.
- h is linear if all triples (f, g, h) associate.
- b is (semantically) negative if all morphisms into b are thunkable. Equivalently, if all morphisms out of b are intermediate.
- c is (semantically) positive when all morphisms out of c are linear. Equivalently, if all morphisms into c are intermediate.

These are depicted pictorially in Figure 1.

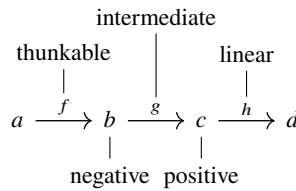


Fig. 1. The triple (f, g, h) associates if any of the labelled properties holds.

⁴ The hom-set law is orthogonal to matters of size, and says that $f =_{\mathcal{M}[[a, b]]} g$ is a proposition.

Definition 4 ([16]). Let $\mathcal{M}$ be a unital magmoid. The following are subcategories of $\mathcal{M}$ – that is, unital sub-magmoids of $\mathcal{M}$ which are associative.

- The linear subcategory $\mathcal{M}_l$ of linear morphisms.
- The thunkable subcategory $\mathcal{M}_t$ of thunkable morphisms.
- The intermediate subcategory $\mathcal{M}_i$ of intermediate morphisms.
- The (semantically) negative subcategory $\mathcal{M}^\ominus$ of semantically negative objects.
- The (semantically) positive subcategory $\mathcal{M}^\oplus$ of semantically positive objects.
- Intersections of the above, notably:
 - The linear-and-thunkable subcategory $\mathcal{M}_{lt}$, of linear and thunkable morphisms.
 - The linear-and-thunkable-and-intermediate subcategory $\mathcal{M}_{lti}$.
 - The negative-linear subcategory $\mathcal{M}_l^\ominus$.
 - The positive-thunkable subcategory $\mathcal{M}_t^\oplus$.

Notation 1. We write the sub/super-scripts from Definition 4 onto the binary operators of morphisms ($\rightarrow$) or isomorphisms ($\cong$) to denote that the morphism or isomorphism (and its inverse) lies in the corresponding subcategory of the unital magmoid.

Convention 2. Phrasing such as "linear-and-thunkable isomorphism" has the meaning suggested by Notation 1. That is, a linear-and-thunkable isomorphism $f : a \cong_{lt} b$ has that both $f : a \rightarrow_{lt} b$ and its inverse $f^{-1} : b \rightarrow_{lt} a$ are linear as well as thunkable.

3.1 Univalence

Internal Univalence The correct notion of isomorphism in a unital magmoid is a linear-and-thunkable-and-intermediate one. All three are required for good behaviour:

Remark 1. Linearity of $p : a \cong_l b$ corresponds to naturality of post-composition with p , as $(- \circ p) : \mathcal{M}[-, a] \simeq \mathcal{M}[-, b]$. (Despite $\mathcal{M}[-, a]$ not generally satisfying functoriality, "natural transformations" remain perfectly well-defined on it.) Thunkability is dual. Intermediateness is required to make $(f \circ p) \circ (p^{-1} \circ g) = f \circ g$ hold, as it is not implied by linearity and thunkability.⁵ Correctness may also be justified using general techniques: linear-and-thunkable-and-intermediate isomorphisms $a \cong_{lti} b$ correspond to the indiscernibilities $a \asymp b$ of Ahrens *et al.* [3]. We therefore define:

Definition 5 ([2]). A category $\mathcal{C}$ is univalent if the canonical map $(a = b) \rightarrow (a \cong b)$, given by $\text{id_to_iso refl}_a \stackrel{\text{def}}{=} \text{id}_a$, is an equivalence for all $a, b : \mathcal{C}$.

Definition 6. A unital magmoid $\mathcal{M}$ is univalent if the linear-and-thunkable-and-intermediate subcategory $\mathcal{M}_{lti}$ is univalent, so that $(a = b) \simeq (a \cong_{lti} b)$ for all $a, b : \mathcal{C}$.

⁵ Consider the free unital magmoid **Lini** generated by morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ and a linear-and-thunkable isomorphism $p : b \cong_{lt} d$, formalized as `lini`. It may be constructed explicitly with finite types such that $|\mathbf{Lini}[a, c]| = 2$ and all other hom-sets are singletons or empty.

External Univalence As is typical in category theory, we are not only concerned with unital magmoids themselves, but also with maps and equivalences between them. We use the term "functor" for maps between unital magmoids with the same definition as for categories. As usual, a functor whose morphism part $F_1 : \mathcal{N}[[a, b]] \rightarrow \mathcal{M}[[F_0 a, F_0 b]]$ is an equivalence is called *fully faithful*. A fully faithful functor is further called an *isomorphism*, written $F : \mathcal{N} \cong \mathcal{M}$, if its object part $F_0 : \text{ob } \mathcal{N} \rightarrow \text{ob } \mathcal{M}$ is also an equivalence of types. Isomorphisms render the respective categories indistinguishable to the type theory. For example, we may derive the following:

Theorem 1. *If $\mathcal{C} \cong \mathcal{D}$ for categories, then $\mathcal{C}$ is univalent if and only if $\mathcal{D}$ is.* $\square$

Isomorphism of categories is not generally as easy to prove as adjoint equivalence. Between univalent categories, however, the notions coincide. Univalent categories are even stronger, in fact. It is a theorem, without the axiom of choice, that for a univalent category $\mathcal{C}$ and a category $\mathcal{D}$, a fully-faithful and essentially surjective functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an adjoint equivalence [2]. The analogous statement in set-theoretic foundations amounts to the axiom of choice.

We obtain a similar result for unital magmoids in Theorems 2 and 3. In the following, let $\mathcal{N}$ and $\mathcal{M}$ be unital magmoids and $F : \mathcal{N} \rightarrow \mathcal{M}$ a functor.

Definition 7. *For $b : \mathcal{M}$, the type $\text{wfib}_F(b) \stackrel{\text{def}}{=} \sum_{a:\mathcal{N}} Fa \cong_{\text{li}} b$ is called the (essential) fibre of F over b . The functor F is essentially surjective when all its fibres are inhabited, and split essentially surjective when all its fibres have a chosen element.*

Definition 8. *A fully-faithful functor is called a weak equivalence if it is essentially surjective, and a strong equivalence if it is split essentially surjective.*

Lemma 1. *If $\mathcal{N}$ is univalent and F fully faithful, then all fibres of F are propositions.*

Proof. Elements $\langle a, Fa \stackrel{e}{\cong}_{\text{li}} b \rangle$ and $\langle a', Fa' \stackrel{e'}{\cong}_{\text{li}} b \rangle$ of $\text{wfib}_F(b)$ can be identified by reflecting $Fa \stackrel{e}{\cong}_{\text{li}} b \cong_{\text{li}} Fa' \stackrel{e'}{\cong}_{\text{li}} b$ in F and using univalence of $\mathcal{N}$ to obtain $p : a = a'$ and $F(\text{id_to_iso } p) = e \circ e'$. Induction on p exposes $F(\text{id}_a) = e \circ e'^{-1}$ and thus $e = e'$. $\square$

Theorem 2. *If $\mathcal{N}$ is univalent and F a weak equivalence, F is split essentially surjective.*

Proof. By Lemma 1 and unique choice. $\square$

Theorem 3. *Weak equivalences between univalent unital magmoids are isomorphisms.*

Proof. The object part is injective by Lemma 1 and surjective by univalence. $\square$

4 Duploids

In this section we introduce *duploids*: unital magmoids equipped with *polarity shifts*. In subsection 4.1, we introduce two variants of duploid: a more recent definition that treats polarities as a property [5,12], and one closer to the original that treats polarities as structure [15,16]. We coin the terms *single-sorted duploid* and *split duploid* for the two. The purpose of considering both in this paper is that they have different univalence conditions, and thus different constructions of the Duploid from an Adjunction. In subsection 4.2 we define what it means for a duploid to be univalent, and show that it makes polarity shifts a property rather than structure.

4.1 Definitions and Properties

Definition 9 ([16,5,12]). A (single-sorted) preduploid is a unital magmoid such that all objects are semantically negative or positive.

A (single-sorted) duploid is a preduploid $\mathcal{D}$ equipped with polarity shifts:

1. Negative shifts. For every object $a : \mathcal{D}$,
 - (a) its upshift, a semantically negative object written $\uparrow a : \mathcal{D}^\ominus$,
 - (b) a linear morphism $\text{force}_a : \uparrow a \rightarrow_l a$,
 - (c) a linear-and-thunkable inverse for force_a called $\text{delay}_a : a \rightarrow_{lt} \uparrow a$.
2. Positive shifts. Dually, for every object $a : \mathcal{D}$,
 - (a) its downshift, a semantically positive object written $\downarrow a : \mathcal{D}^\oplus$,
 - (b) a thunkable morphism $\text{wrap}_a : a \rightarrow \downarrow a$,
 - (c) a linear-and-thunkable inverse for wrap_a called $\text{unwrap}_a : \downarrow a \rightarrow_{lt} a$.

Definition 10. A split preduploid is a unital magmoid $\mathcal{D}$ equipped with a chosen polarity mapping $\omega : \text{ob } \mathcal{D} \rightarrow \{\ominus, \oplus\}$. We say that an object a is syntactically negative if $\omega(a) = \ominus$ and syntactically positive if $\omega(a) = \oplus$, and further require that every syntactically negative/positive object is also **semantically** negative/positive.

A split duploid is a split preduploid equipped with polarity shifts, such that the polarity mapping respects the shifts. That is, all $\uparrow a / \downarrow a$ must be syntactically negative/positive.

Remark 2. Every split preduploid is evidently a single-sorted preduploid. Conversely, the law of excluded middle gives a polarity mapping for any preduploid. For example, by setting $\omega(a) \stackrel{\text{def}}{=} \ominus$ if a is semantically negative, and $\omega(a) \stackrel{\text{def}}{=} \oplus$ otherwise. The law of excluded middle does not help for defining a polarity mapping for duploids, as ω defined as such will not respect the polarity shifts in general.

Example 1. A category C can be viewed as a duploid with polarity shifts $\langle \uparrow a, \text{force}_a \rangle \stackrel{\text{def}}{=} \langle a, \text{id}_a \rangle$ and $\langle \downarrow a, \text{wrap}_a \rangle \stackrel{\text{def}}{=} \langle a, \text{id}_a \rangle$. If an object $a : C$ exists, then C has no definable shift-respecting polarity mapping.

Definition 11. Let $\mathcal{D}$ be a split preduploid with polarity mapping $\omega : \text{ob } \mathcal{D} \rightarrow \{\ominus, \oplus\}$. Extending Definition 4, there are the following subcategories of $\mathcal{D}$.

- The syntactically negative subcategory $\mathcal{D}^{\ominus\omega}$ of syntactically negative objects.
- The syntactically positive subcategory $\mathcal{D}^{\oplus\omega}$ of syntactically positive objects.
- Intersections with those of Definition 4, notably:
 - The syntactically negative-linear subcategory $\mathcal{D}_l^{\ominus\omega}$.
 - The syntactically positive-thunkable subcategory $\mathcal{D}_t^{\oplus\omega}$.

Remark 3. Definition 9 makes the polarities in a preduploid merely a property on unital magmoids, rather than structure as Munch-Maccagnoni [16] first defined it. The original *two-sorted* duploids of *op. cit.* are precisely the split duploids $\mathcal{D}$ where $\langle \uparrow a, \text{force}_a \rangle \equiv \langle a, \text{id}_a \rangle$ for syntactically negative $a : \mathcal{D}$, and $\langle \downarrow b, \text{wrap}_b \rangle \equiv \langle b, \text{id}_b \rangle$ for syntactically positive $b : \mathcal{D}$. Any split duploid's shifts can be strictified to this form.

We note the following characterizations of polarity in a duploid.

Theorem 4 ([16, Proposition 13]). *In a duploid, a morphism $f : a \rightarrow b$ is thunkable if and only if the triple $a \xrightarrow{f} b \xrightarrow{\text{delay}_b} {}_{lt} \uparrow b \xrightarrow{\text{wrap}_{\uparrow b}} {}_{ti} \uparrow \downarrow b$ associates. Dually, $f : a \rightarrow b$ is linear if and only if the triple $\uparrow \downarrow a \xrightarrow{\text{force}_{\downarrow a}} {}_{li} \downarrow a \xrightarrow{\text{unwrap}_a} {}_{lt} a \xrightarrow{f} b$ associates.* $\square$

Theorem 5. *If $p : a \cong_{lt} b$ is a linear-and-thunkable isomorphism in a unital magmoid, then a is positive if and only if b is positive, and a is negative if and only if b is.* $\square$

Corollary 1. *An object $a : \mathcal{D}$ in a duploid $\mathcal{D}$ is negative if and only if $\text{force}_a : \uparrow a \rightarrow_l a$ is thunkable. Dually, $a : \mathcal{D}$ is positive if and only if $\text{wrap}_a : a \rightarrow_t \downarrow a$ is linear.* $\square$

4.2 Univalence

Since single-sorted preduploids are just unital magmoids with an extra property, their univalence condition should be the same. However, we can simplify the notion of isomorphism used. We also find that univalence of preduploids has a useful characterization in terms of the polarized subcategories.

Theorem 6. *If an object a in a unital magmoid is negative or positive, then all linear-and-thunkable isomorphisms $p : a \cong_{lt} b$ are intermediate.*

Proof. By Theorem 5, a and b share a polarity. Intermediateness of p is immediate. $\square$

Corollary 2. *Linear-and-thunkable isomorphisms in preduploids are intermediate.* $\square$

Definition 12. *A duploid or preduploid $\mathcal{D}$ is univalent (as a single-sorted (pre)duploid) if its linear-and-thunkable subcategory $\mathcal{D}_{lt}$ is univalent.*

Remark 4. Definition 12 coincides with univalence as a unital magmoid by Corollary 2.

We now obtain a characterization of univalence for preduploids in terms of its polarized subcategories. We shall use a specialization of Rijke's *fundamental theorem of identity types* [17] for the proof.

Definition 13. *For an object $a : C$ in a category C , the type $\{a\}_C^{\Rightarrow} \stackrel{\text{def}}{=} \sum_{b:C} a \cong b$ is called the outward fan of a , and $\{a\}_C^{\Leftarrow} \stackrel{\text{def}}{=} \sum_{b:C} b \cong a$ is called the inward fan of a .*

Theorem 7 ([17, Ch. 11]). *A category C is univalent if and only if all its outward (dually, inward) fans are propositions.* $\square$

Theorem 8. *A preduploid $\mathcal{D}$ is univalent if and only if $\mathcal{D}_l^{\ominus}$ and $\mathcal{D}_t^{\oplus}$ are univalent.*

Proof. The "only if" direction follows from fully-faithfulness of the inclusion functors $\mathcal{D}_l^{\ominus} \hookrightarrow \mathcal{D}_{lt} \hookleftarrow \mathcal{D}_t^{\oplus}$. For the "if" direction, we apply Theorem 7 and show that every fan $\{a\}_{\mathcal{D}_{lt}}^{\Rightarrow}$ is a proposition. Using the preduploid property, if $a : \mathcal{D}$ is negative, the fan $\{a\}_{\mathcal{D}_{lt}}^{\Rightarrow}$ is equivalent to the fan $\{a\}_{\mathcal{D}_l^{\ominus}}^{\Rightarrow}$ by Theorem 5. The latter is a proposition by assumption. Dually, $\{a\}_{\mathcal{D}_{lt}}^{\Leftarrow} \simeq \{a\}_{\mathcal{D}_t^{\oplus}}^{\Leftarrow}$ if a is positive. $\square$

For split (pre)duploids, Definition 12 is incorrect in general, due to the fact that objects may be both negative and positive semantically, but not syntactically. Instead, we use a syntactic variant of Theorem 8 as the definition of univalence for split duploids.

Theorem 9. *If a split duploid $\mathcal{D}$ has an object $a : \mathcal{D}$ which is both semantically negative and semantically positive, then the underlying single-sorted duploid of $\mathcal{D}$ is not univalent.*

Proof. Univalence applied to $\uparrow a \cong_{\text{it}} \downarrow a$ would imply $\ominus = \omega(\uparrow a) = \omega(\downarrow a) = \oplus$. $\square$

Definition 14. *A split duploid or preduploid $\mathcal{D}$ is univalent (as a split duploid) if both its syntactically negative-linear subcategory $\mathcal{D}_l^{\ominus\omega}$ and its syntactically positive-thunkable subcategory $\mathcal{D}_t^{\oplus\omega}$ are univalent.*

Remark 5. Definition 14 coincides with the indiscernibilities derived for two-sorted duploids by Ahrens *et al.* [3, Example 13.5].

Objects in category theory are often unique up to canonical isomorphism. In these cases, univalent categories make them literally unique; that is, propositions. The same holds for unital magmoids, and polarity shifts are one such example.

Lemma 2. *For a univalent unital magmoid $\mathcal{M}$ and object $a : \mathcal{M}$, we have that the types $\sum_{b:\mathcal{M}^{\ominus}} (b \cong_l a)$ and $\sum_{b:\mathcal{M}^{\oplus}} (a \cong_t b)$ are propositions.*

Proof. Let $\langle b, b \cong_l^e a \rangle$ and $\langle b', b' \cong_l^{e'} a \rangle$ be of the former type. Then by Theorem 7 we have $\langle b, e \circ e'^{-1} \rangle =_{\{b'\}_{\mathcal{D}_{\text{iti}}}} \langle b', \text{id}_{b'} \rangle$. It follows that $b = b'$ and $e = e'$. $\square$

Theorem 10. *For a univalent preduploid $\mathcal{D}$, the type " $\mathcal{D}$ is a duploid" is a proposition.*

Proof. The data $\langle \uparrow a, \text{force}_a \rangle$ and $\langle \downarrow a, \text{wrap}_a \rangle$ are of the types in Lemma 2. $\square$

Remark 6. Lemma 2 suggests a definition of representability for unital magmoids. In unital magmoids, natural isomorphisms $\mathcal{M}[-, a] \cong \mathcal{M}[-, b]$ are linear isomorphisms $a \cong_l b$ (by Remark 1). Thus a non-functorial map $F : \mathcal{M}^{\text{op}} \rightarrow \mathbf{Set}$ is *representable* if there is a negative object $b : \mathcal{D}^{\ominus}$ satisfying $\mathcal{M}[-, b] \cong F$, and dually with positive objects. In that sense, the polarity shift structure of duploids amounts to saying that all $\mathcal{D}[-, b]$ and $\mathcal{D}[a, -]$ are representable, by $\langle \uparrow b, \text{force}_b \rangle$ and $\langle \downarrow a, \text{wrap}_a \rangle$ respectively.

5 Duploids from Adjunctions

Duploids are related to adjunctions: every adjunction gives rise to a duploid, and every duploid is equivalent to one that arises from an adjunction. The duploid arising from an adjunction was first constructed by Munch-Maccagnoni [16] as a two-sorted duploid, which (by Theorem 9) fails to be univalent as a single-sorted duploid in general. We give this construction the name *oblique duploid*, and seek to give an equivalent single-sorted construction that can be univalent. We shall explain the new construction, the *envelope duploid*, in subsections 5.1–5.3. Finally, in subsection 5.4 we shall prove its univalence.

5.1 Construction

Morphisms Recall that an adjunction $L \dashv R : \mathcal{N} \rightarrow \mathcal{P}$ may be viewed as an isomorphism:

$$\varphi_{p,n} : \mathcal{N}[\llbracket Lp, n \rrbracket] \simeq \mathcal{P}[\llbracket p, Rn \rrbracket] \quad \text{natural in } p : \mathcal{P}^{\text{op}} \text{ and } n : \mathcal{N}.$$

Like Levy [9], we call the morphisms on either side of $\varphi_{p,n}$ the *oblique morphisms*, written $O[\llbracket p, n \rrbracket]$. Rather than making an arbitrary choice of the side, however, we use the following symmetric definition:

Definition 15. An oblique morphism $f : O[\llbracket p, n \rrbracket]$ consists of a negative mate $f^{\flat} : \mathcal{N}[\llbracket Lp, n \rrbracket]$ and a positive mate $f^{\sharp} : \mathcal{P}[\llbracket p, Rn \rrbracket]$, satisfying $\varphi_{p,n}(f^{\flat}) = f^{\sharp}$. The family $O[\llbracket p, n \rrbracket]$ extends to a functor $O : \mathcal{P}^{\text{op}} \times \mathcal{N} \rightarrow \mathbf{Set}$ by pre/post-composition in $\mathcal{P}$ and $\mathcal{N}$.

Objects Our morphisms shall be given by oblique morphisms. But what should the objects be? A first approach may be to use objects from either $\mathcal{N}$ or $\mathcal{P}$.

Definition 16. An oblique object $a : O[L \dashv R]$ is either $\iota_O^{\ominus} n$ for $n : \mathcal{N}$ or $\iota_O^{\oplus} p$ for $p : \mathcal{P}$. For an oblique object a , define the negative and positive projections, $\pi_O^{\ominus} a : \mathcal{N}$ and $\pi_O^{\oplus} a : \mathcal{P}$ in the obvious way, as

$$\begin{aligned} \pi_O^{\ominus}(\iota_O^{\ominus} n) &\stackrel{\text{def}}{=} n & \pi_O^{\oplus}(\iota_O^{\ominus} n) &\stackrel{\text{def}}{=} Rn \\ \pi_O^{\ominus}(\iota_O^{\oplus} p) &\stackrel{\text{def}}{=} Lp & \pi_O^{\oplus}(\iota_O^{\oplus} p) &\stackrel{\text{def}}{=} p. \end{aligned}$$

The morphisms between oblique objects a and b are defined as $O[\pi_O^{\oplus} a, \pi_O^{\ominus} b]$. This is the start of the definition of the oblique duploid.

In light of Theorem 9, this approach has an unfortunate deficiency: it creates extra structure by making a polarity mapping $\omega(\iota_O^{\varepsilon}(x)) \stackrel{\text{def}}{=} \varepsilon : \{\ominus, \oplus\}$ definable.⁶ For an alternative definition without a definable polarity mapping, we must use different objects.

Let us consider what the objects need. Reasoning backwards, to define morphisms $a \rightarrow b$ in terms of $O[\llbracket p, n \rrbracket]$, we shall need to associate some $p : \mathcal{P}$ and some $n : \mathcal{N}$ to all objects a, b . Next, in order to uniformly define identities, it stands to reason that the $p : \mathcal{P}$ and $n : \mathcal{N}$ associated to an object a ought to be related by a morphism $O[\llbracket p, n \rrbracket]$ that will become id_a . We call the data $p : \mathcal{P}, n : \mathcal{N}$ and $f : O[\llbracket p, n \rrbracket]$ an *envelope pre-object*; f will later need an extra property to behave like an identity.

Definition 17. An envelope pre-object a in $L \dashv R : \mathcal{N} \rightarrow \mathcal{P}$ is a triple of an object $\pi_{\mathcal{E}}^{\ominus} a : \mathcal{N}$, an object $\pi_{\mathcal{E}}^{\oplus} a : \mathcal{P}$, and a morphism $\text{cross}_{\mathcal{E}} a : O[\pi_{\mathcal{E}}^{\oplus} a, \pi_{\mathcal{E}}^{\ominus} a]$. We may write an envelope pre-object a as simply the diagram of its oblique morphism:

$$\pi_{\mathcal{E}}^{\oplus} a \xrightarrow{\text{cross}_{\mathcal{E}} a} \pi_{\mathcal{E}}^{\ominus} a.$$

The morphisms between two envelope pre-objects a and b , are given by $O[\pi_{\mathcal{E}}^{\oplus} a, \pi_{\mathcal{E}}^{\ominus} b]$.

⁶ The "extra structure" is apparent if the type $\mathcal{N} \amalg \mathcal{P}$ is expressed as $\sum_{\varepsilon : \{\ominus, \oplus\}} E(\varepsilon)$ for $E(\ominus) \stackrel{\text{def}}{=} \text{ob } \mathcal{N}$ and $E(\oplus) \stackrel{\text{def}}{=} \text{ob } \mathcal{P}$. From this perspective, the polarity is an extra data field!

Notation 2. The operator cross should be read with a higher precedence than $\flat$ and $\sharp$. That is, all three of $(\text{cross } a)^\flat$, $\text{cross}(a)^\flat$, and $\text{cross } a^\flat$ mean the same thing.

Definition 18. To tie envelope pre-objects back to the oblique objects, we may associate to each oblique object $a : \mathcal{O}[L \dashv R]$ an envelope pre-object $\widehat{a}$:

$$\left(\pi_{\mathcal{E}}^{\oplus} \widehat{a} \xrightarrow{\text{cross}_{\mathcal{E}} \widehat{a}} \pi_{\mathcal{E}}^{\ominus} \widehat{a} \right) \stackrel{\text{def}}{=} \left(\pi_{\mathcal{O}}^{\oplus} a \xrightarrow{\text{cross}_{\mathcal{O}} a} \pi_{\mathcal{O}}^{\ominus} a \right),$$

where $\text{cross}_{\mathcal{O}} a : \mathcal{O}[\pi_{\mathcal{O}}^{\oplus} a, \pi_{\mathcal{O}}^{\ominus} a]$ is defined as

$$\begin{aligned} \text{cross}_{\mathcal{O}}(\iota_{\mathcal{O}}^{\ominus} n)^{\flat} &\stackrel{\text{def}}{=} LRn \xrightarrow{\epsilon_n} n & \text{cross}_{\mathcal{O}}(\iota_{\mathcal{O}}^{\oplus} p)^{\flat} &\stackrel{\text{def}}{=} Rn \xrightarrow{\text{id}_{Rn}} Rn \\ \text{cross}_{\mathcal{O}}(\iota_{\mathcal{O}}^{\ominus} n)^{\sharp} &\stackrel{\text{def}}{=} Lp \xrightarrow{\text{id}_{Lp}} Lp & \text{cross}_{\mathcal{O}}(\iota_{\mathcal{O}}^{\oplus} p)^{\sharp} &\stackrel{\text{def}}{=} p \xrightarrow{\eta_p} RLP. \end{aligned}$$

Turning the definition of $\widehat{a}$ around, note that oblique objects are precisely the envelope pre-objects b for which either $\text{cross}_{\mathcal{E}} b^{\sharp}$ or $\text{cross}_{\mathcal{E}} b^{\flat}$ is the identity. Thus, to lose the polarity mapping structure, we could propositionally truncate away the aforementioned condition. Of course, internal to the type theory, we cannot compare an arbitrary morphism to the identity: $\text{cross}_{\mathcal{E}} b^{\flat} =_{\mathcal{N}[L(\pi_{\mathcal{E}}^{\oplus} b), ??]} \text{id}_{L(\pi_{\mathcal{E}}^{\oplus} b)}$ is ill-typed! Instead, we ask for $\text{cross}_{\mathcal{E}} b^{\sharp}$ or $\text{cross}_{\mathcal{E}} b^{\flat}$ to be an *isomorphism* – an inverse for either is called a *polarization choice*. This is the inspiration behind the envelope duploid.

Definition 19. A polarization choice for an envelope pre-object b is an inverse for either mate of $\text{cross}_{\mathcal{E}} b$. That is, either an inverse $(\text{cross}_{\mathcal{E}} b^{\sharp})^{-1} : R(\pi_{\mathcal{E}}^{\ominus} b) \rightarrow \pi_{\mathcal{E}}^{\oplus} b$ for $\text{cross}_{\mathcal{E}} b^{\sharp}$ – a negative polarization choice; or one $(\text{cross}_{\mathcal{E}} b^{\flat})^{-1} : \pi_{\mathcal{E}}^{\ominus} b \rightarrow L(\pi_{\mathcal{E}}^{\oplus} b)$ for $\text{cross}_{\mathcal{E}} b^{\flat}$ – a positive polarization choice.

Definition 20. An envelope object $a : \mathcal{E}[L \dashv R]$ is an envelope pre-object such that there exists a polarization choice for a . That is, $\text{cross}_{\mathcal{E}} a^{\sharp}$ or $\text{cross}_{\mathcal{E}} a^{\flat}$ is an isomorphism.

Remark 7. Beware Convention 1: the polarization choice for each envelope object is under propositional truncation. Importantly, it does not partake in identifications of envelope objects, and may only be obtained constructively if the choice is irrelevant.

Example 2. Every oblique object is clearly an envelope object: each $\iota_{\mathcal{E}}^{\ominus} n \stackrel{\text{def}}{=} \widehat{\iota_{\mathcal{O}}^{\ominus} n}$ has a negative polarization choice, and each $\iota_{\mathcal{E}}^{\oplus} p \stackrel{\text{def}}{=} \widehat{\iota_{\mathcal{O}}^{\oplus} p}$ a positive one.

Notation 3. We shall freely omit the subscripts $\mathcal{E}$ from $\iota_{\mathcal{E}}^{\oplus}$, $\iota_{\mathcal{E}}^{\ominus}$, $\pi_{\mathcal{E}}^{\oplus}$, $\pi_{\mathcal{E}}^{\ominus}$, and $\text{cross}_{\mathcal{E}}$, and likewise for the $\mathcal{O}$ variants. The conflation between $\mathcal{E}$ and $\mathcal{O}$ variants is intentional, to encourage viewing oblique objects as envelope objects.

Composition With the objects well-defined, we now seek to define the composite $f \circ g : \mathcal{O}[\pi^{\oplus} a, \pi^{\ominus} c]$ of oblique morphisms $f : \mathcal{O}[\pi^{\oplus} a, \pi^{\ominus} b]$ and $g : \mathcal{O}[\pi^{\oplus} b, \pi^{\ominus} c]$ between envelope objects. There is no immediate way of composing f and g in either $\mathcal{N}$ or $\mathcal{P}$. Instead, as for the original oblique duploid, the composite will depend on the polarity of the intermediate object [16]. What does the polarity give us?

Definition 21. If an envelope pre-object b has a negative polarity choice, there exists for all $g : O[p, \pi^\ominus b]$ a unique $g^\sharp \stackrel{\text{def}}{=} g^\# \circ (\text{cross } b^\#)^{-1} : \mathcal{P}[p, \pi^\oplus b]$ with $g^\sharp \circ \text{cross } b^\# = g^\#$:

$$\begin{array}{ccc} p & \xrightarrow{g^\sharp} & \\ \exists! g^\sharp \downarrow & \searrow & \\ \pi^\oplus b & \xrightarrow{\text{cross } b^\#} & R(\pi^\ominus b) \end{array} \quad O[p, \pi^\ominus b] \simeq \mathcal{P}[p, \pi^\oplus b] \quad \text{if } b \text{ has a negative choice.}$$

Dually, if an envelope pre-object b has a positive polarity choice, there exists for all $f : O[\pi^\oplus b, n]$ a unique $f^\flat \stackrel{\text{def}}{=} (\text{cross } b^\flat)^{-1} \circ f^\flat : \mathcal{N}[\pi^\ominus b, n]$ with $\text{cross } b^\flat \circ f^\flat = f$:

$$\begin{array}{ccc} L(\pi^\oplus b) & \xrightarrow{\text{cross } b^\flat} & \pi^\ominus b \\ & \searrow f^\flat & \downarrow \exists! f^\flat \\ & & n \end{array} \quad O[\pi^\oplus b, n] \simeq \mathcal{N}[\pi^\ominus b, n] \quad \text{if } b \text{ has positive choice.}$$

We may now resolve the composite $f \circ g : O[\pi^\oplus a, \pi^\ominus c]$.

Definition 22. Let a, b , and c be envelope pre-objects in $L \dashv R$, and let $f : O[\pi^\oplus a, \pi^\ominus b]$ and $g : O[\pi^\oplus b, \pi^\ominus c]$ be oblique morphisms between them. Given a negative polarity choice for b , define $f \circ g \stackrel{\text{def}}{=} O[f^\sharp, \pi^\ominus c](g)$, whose mates are depicted:

$$\begin{array}{ccc} L(\pi^\oplus a) & \xrightarrow{f^\flat} & L(\pi^\oplus b) \\ \downarrow L(f^\sharp) & & \downarrow \text{cross } b^\flat \\ L(\pi^\oplus b) & \xrightarrow{\text{cross } b^\flat} & \pi^\ominus b \\ & \searrow g^\flat & \\ & & \pi^\ominus c \end{array} \quad \begin{array}{ccc} \pi^\oplus a & \xrightarrow{f^\sharp} & \pi^\oplus b \\ \downarrow f^\sharp & & \downarrow \text{cross } b^\sharp \\ \pi^\oplus b & \xrightarrow{\text{cross } b^\sharp} & R(\pi^\ominus b) \\ & \searrow g^\sharp & \\ & & R(\pi^\ominus c) \end{array} \quad (1)$$

Dually, given a positive polarity choice for b , define $f \circ g \stackrel{\text{def}}{=} O[\pi^\oplus a, g^\flat](f)$:

$$\begin{array}{ccc} L(\pi^\oplus a) & \xrightarrow{f^\flat} & L(\pi^\oplus b) \\ & \searrow g^\flat & \downarrow g^\flat \\ & & \pi^\ominus c \end{array} \quad \begin{array}{ccc} \pi^\oplus a & \xrightarrow{f^\sharp} & \pi^\oplus b \\ & \searrow g^\sharp & \downarrow R(g^\flat) \\ & & R(\pi^\ominus c) \end{array} \quad (2)$$

Remark 8. If b in $a \xrightarrow{f} b \xrightarrow{g} c$ has both a negative and a positive polarization choice, then pasting of Diagrams 1 and 2 reveals that both ways of composing $f \circ g : a \rightarrow c$ in Definition 22 are equal. This is enough to justify (with respect to Remark 7) using the polarisation choice that exists for each envelope object.

We now have the underlying unital magmoid of the envelope duploid:

Definition 23. The envelope duploid-to-be $\mathcal{E}[L \dashv R]$ over $L \dashv R : \mathcal{N} \rightarrow \mathcal{P}$ has:

1. The objects $\text{ob } \mathcal{E}[L \dashv R]$ are envelope objects (Definition 20).
2. Morphisms $\mathcal{E}[L \dashv R][a, b]$ are the oblique morphisms $O[\pi^\oplus a, \pi^\ominus b]$.
3. Composition is as in Definition 22, using the intermediate object's polarity choice.
4. For an object $a : \mathcal{E}[L \dashv R]$, its identity is given by $\text{id}_a \stackrel{\text{def}}{=} \text{cross } a$.
5. The universal property of Definition 21 is precisely that composition is unital.

We may also define the oblique duploid by reference to the envelope duploid.

Definition 24. The oblique duploid-to-be $\mathcal{E}[L \dashv R]$ over $L \dashv R$ has the following:

1. The objects $\text{ob } O[L \dashv R] \stackrel{\text{def}}{=} \mathcal{N} \amalg \mathcal{P}$ are oblique objects (Definition 16).
2. Morphisms, composition, and identities are those of the envelope duploid, where each oblique object is viewed as an envelope object (Definition 18).

Polarities and Shifts We now seek to show that the envelope structure is a duploid. First, we must show it to be a preduploid: that each object is semantically negative or positive.

Lemma 3. Objects in the envelope duploid-to-be $\mathcal{E}[L \dashv R]$ with a negative polarization choice are negative, and those with a positive polarization choice are positive. $\square$

Corollary 3. Every object of the envelope duploid-to-be is negative or positive based on its associated polarization choice. Thus, $\mathcal{E}[L \dashv R]$ is a single-sorted preduploid. $\square$

Corollary 4. The mapping $\omega(\iota_O^\mathcal{E}(x)) \stackrel{\text{def}}{=} \varepsilon : \{\ominus, \oplus\}$ is a valid polarity mapping for the oblique duploid-to-be. Thus, $O[L \dashv R]$ is a split preduploid. $\square$

We now seek to define polarity shifts. Since each envelope object comes associated with an object from each of the negative and positive categories, it is straightforward to find for each a semantically negative or positive object in the envelope duploid.

Definition 25. Let $a : \mathcal{E}[L \dashv R]$ be an envelope object. Define its polarity shifts as:

$$\begin{aligned} \uparrow a &\stackrel{\text{def}}{=} \iota^\ominus(\pi^\ominus a) : \mathcal{E}[L \dashv R] & \Downarrow a &\stackrel{\text{def}}{=} \iota^\oplus(\pi^\oplus a) : \mathcal{E}[L \dashv R] \\ \text{force}_a &\stackrel{\text{def}}{=} \text{cross}(\uparrow a) : \mathcal{E}[L \dashv R][\uparrow a, a] & \text{wrap}_a &\stackrel{\text{def}}{=} \text{cross}(\Downarrow a) : \mathcal{E}[L \dashv R][a, \Downarrow a] \\ \text{delay}_a &\stackrel{\text{def}}{=} \text{cross } a : \mathcal{E}[L \dashv R][a, \uparrow a] & \text{unwrap}_a &\stackrel{\text{def}}{=} \text{cross } a : \mathcal{E}[L \dashv R][\Downarrow a, a]. \end{aligned}$$

We leave it to the reader to verify the axioms from Definition 9.

This completes the construction of the envelope and oblique duploids:

Theorem 11. The envelope duploid $\mathcal{E}[L \dashv R]$ of Definition 23 is a single-sorted duploid, with polarity shifts given by Definition 25. $\square$

Theorem 12. The oblique duploid $O[L \dashv R]$ of Definition 24 is a split duploid, with polarity shifts given by Definition 25. $\square$

5.2 Polarities in the Envelope Duploid

Before we prove univalence of the construction from subsection 5.1, we note some important theorems, starting with characterizations of linearity, thunkability, and polarities following from Theorem 4. In the following, let $L \dashv_{(\eta, \epsilon)} R : \mathcal{N} \rightarrow \mathcal{P}$ be an adjunction.

Lemma 4 ([16, Proposition 14]). *A morphism $f : a \rightarrow \iota_{\mathcal{E}}^{\oplus} p$ in $\mathcal{E}[L \dashv R]$ is thunkable if and only if $f^{\sharp} \circ RL\eta_p = f^{\sharp} \circ \eta_{RLp}$ holds. Dually, a morphism $g : \iota_{\mathcal{E}}^{\ominus} n \rightarrow b$ in $\mathcal{E}[L \dashv R]$ is linear if and only if $LR\epsilon_n \circ g^{\flat} = \epsilon_{LRn} \circ g^{\flat}$ holds. $\square$*

Theorem 13. *For all $p : \mathcal{P}$, the object $\iota^{\oplus} p$ is negative in $\mathcal{E}[L \dashv R]$ if and only if the equation $RL\eta_p = \eta_{RLp}$ holds. Dually, for $n : \mathcal{N}$, the object $\iota^{\ominus} n$ is positive in $\mathcal{E}[L \dashv R]$ if and only if the equation $LR\epsilon_n = \epsilon_{LRn}$ holds.*

Proof. By, Corollary 1, it is sufficient to check if `force` is linear or `wrap` is thunkable to see that an object is negative or positive. The statement then follows from Lemma 4. $\square$

Remark 9. We may compare the characterization of polarities in Theorem 13 to the statement of semantic polarities from polarization choices in Lemma 3. Specialized to $\iota^{\oplus} p$ and $\iota^{\ominus} n$, Lemma 3 states the following:

For all $p : \mathcal{P}$, the object $\iota^{\oplus} p$ is negative in $\mathcal{E}[L \dashv R]$ if η_p is an isomorphism.
Dually, for $n : \mathcal{N}$, the object $\iota^{\ominus} n$ is positive in $\mathcal{E}[L \dashv R]$ if ϵ_n is an isomorphism.

The converse of Lemma 3 does not hold in general.⁷ However, it does hold for a useful class of adjunctions (see Lemma 10 later).

We may now provide concrete examples of split duploids which are not univalent as single-sorted duploids. The following examples may be checked using Remark 9.

Example 3. Let C be a category with an object $a : C$. Then $\iota^{\ominus} a$ and $\iota^{\oplus} a$ are both semantically negative and positive in the oblique duploid over the identity adjunction $O[\text{Id}_C \dashv \text{Id}_C]$. Thus the latter is not univalent as a single-sorted duploid by Theorem 9.

Example 4. Consider a fixed inhabited set $s : \mathbf{Set}$, and the exponential adjunction $(- \times s) \dashv (-)^s : \mathbf{Set} \rightarrow \mathbf{Set}$. The empty set $\iota^{\ominus} \{\}$ or $\iota^{\oplus} \{\}$ is both semantically negative and positive in $O[(- \times s) \dashv (-)^s]$.

5.3 The Structure Theorem

Every two-sorted duploid is isomorphic to the oblique duploid over some adjunction [15, Proposition II.35]. We have the same for the envelope duploid, but with equivalence of unital magmoids instead of isomorphism.⁸ We also show that the oblique and envelope duploids are equivalent as unital magmoids.

⁷ Consider the unique functor $! : \mathbf{Set} \rightarrow \mathbf{1}$ to the terminal category, and the terminal-object functor $! : \mathbf{1} \rightarrow \mathbf{Set}$ which is its right adjoint. All objects in $\mathcal{E}[! \dashv \text{Id}]$ are (trivially) both negative and positive, but the unit $!_a : a \rightarrow \text{Id}$ is not an isomorphism at e.g. $\{\perp, \top\}$.

⁸ Munch-Maccagnoni [16]’s duploid functors preserve syntactic polarities and linearity/thunkability; all weak equivalences preserve semantic polarities and linearity/thunkability.

Theorem 14. *The duploid $O[L \dashv R]$ is weakly equivalent to $\mathcal{E}[L \dashv R]$.*

Proof. The mapping $a \mapsto \widehat{a}$ on objects $O[L \dashv R] \rightarrow \mathcal{E}[L \dashv R]$ extends trivially to a functor. To see that it is a weak equivalence, observe that every $b : \mathcal{E}[L \dashv R]$ is linearly-and-thunkably isomorphic to $\Uparrow b \equiv \iota^\oplus(\pi^\oplus b)$ or to $\Downarrow b \equiv \iota^\oplus(\pi^\ominus b)$ depending on the polarity choice of b . $\square$

Remark 10 ([16]). The polarity shifts of a duploid $\mathcal{D}$ extend (in general, non-functorially) to morphisms, as $\Uparrow(a \xrightarrow{f} b) \stackrel{\text{def}}{=} \text{force}_a \circ f \circ \text{delay}_b$ and $\Downarrow(a \xrightarrow{f} b) \stackrel{\text{def}}{=} \text{unwrap}_a \circ f \circ \text{wrap}_b$. These are functorial when restricted to $\Downarrow : \mathcal{D}_l^\ominus \rightarrow \mathcal{D}_r^\oplus$ and $\Uparrow : \mathcal{D}_l^\oplus \rightarrow \mathcal{D}_r^\ominus$, which take part in an adjunction $\Uparrow \dashv_{(\eta, \epsilon)} \Downarrow : \mathcal{D}_l^\ominus \rightarrow \mathcal{D}_r^\oplus$ with unit $\eta_p \stackrel{\text{def}}{=} \text{delay}_p \circ \text{wrap}_{\Uparrow p}$ and counit $\epsilon_n \stackrel{\text{def}}{=} \text{force}_{\Downarrow n} \circ \text{unwrap}_n$. We then have:

Theorem 15. *The duploid $\mathcal{D}$ is weakly equivalent to $\mathcal{E}[\Uparrow \dashv \Downarrow : \mathcal{D}_l^\ominus \rightarrow \mathcal{D}_r^\oplus]$.*

Proof. The functor $F : \mathcal{D} \rightarrow \mathcal{E}[\Uparrow \dashv \Downarrow]$ is determined by its action on morphisms

$$\begin{aligned} F_1\left(a \xrightarrow{f} b\right) &: O[\Downarrow a, \Uparrow b] \\ F_1\left(a \xrightarrow{f} b\right)^b &\stackrel{\text{def}}{=} \Uparrow\left(\Downarrow a \xrightarrow{\text{unwrap}_a} a \xrightarrow{f} b\right) \\ F_1\left(a \xrightarrow{f} b\right)^\# &\stackrel{\text{def}}{=} \Downarrow\left(a \xrightarrow{f} b \xrightarrow{\text{delay}_a} \Uparrow b\right). \end{aligned}$$

On objects, we then have $F_0(a) \stackrel{\text{def}}{=} (\Downarrow a \xrightarrow{F_1(\text{id}_a)} \Uparrow a)$. The fibre over each $b : \mathcal{E}[\Uparrow \dashv \Downarrow]$ is inhabited with object part $\pi^\ominus b$ or $\pi^\oplus b$ depending on the polarity of b . If b is negative, we must find an isomorphism $F(\pi^\ominus b) \cong_{lt} b$. We have $\text{force}_b : \Uparrow b \cong_{lt} b$ and, since F can be shown to preserve polarities, $\text{delay}_{F(\pi^\ominus b)} : F(\pi^\ominus b) \cong_{lt} \Uparrow F(\pi^\ominus b)$. We may then compute $\Uparrow F(\pi^\ominus b) \equiv \iota^\ominus(\Uparrow(\pi^\ominus b))$ and $\iota^\ominus(\pi^\ominus b) \equiv \Uparrow b$ to obtain $\iota^\ominus(\text{force}_{\pi^\ominus b}) : \iota^\ominus(\Uparrow(\pi^\ominus b)) \cong_{lt} \iota^\ominus(\pi^\ominus b)$. Ultimately, the isomorphism part is:

$$F(\pi^\ominus b) \stackrel{\text{delay}}{\cong_{lt}} \Uparrow F(\pi^\ominus b) \equiv \iota^\ominus(\Uparrow(\pi^\ominus b)) \stackrel{\iota^\ominus(\text{force})}{\cong_{lt}} \iota^\ominus(\pi^\ominus b) \equiv \Uparrow b \stackrel{\text{force}}{\cong_{lt}} b. \quad \square$$

Remark 11. The equivalences of Theorems 14 and 15 are strong when assuming the law of excluded middle. This is because we can "decide" whether, say, a negative polarization choice for $b : \mathcal{E}[L \dashv R]$ exists, and take an arbitrary bias towards $\pi^\ominus b$ when it does.

5.4 Univalence and the Equalizing Requirement

For the rest of the section, let $L \dashv_{(\eta, \epsilon)} R : \mathcal{N} \rightarrow \mathcal{P}$ be an adjunction. If $\mathcal{N}$ and $\mathcal{P}$ are univalent, when are the oblique and envelope duploids of $L \dashv R$ univalent? We must somehow show that $O[L \dashv R]_l^{\ominus\omega} / O[L \dashv R]_l^{\oplus\omega}$ and $\mathcal{E}[L \dashv R]_l^\ominus / \mathcal{E}[L \dashv R]_l^\oplus$ are univalent. We shall do so using Theorem 1, by exhibiting isomorphisms of categories with $\mathcal{N} / \mathcal{P}$. To do so, we will need an *equalizing requirement* [16, 14].

Lemma 5. (Recall that $O[L \dashv R]_I^{\ominus\omega}$ is the *syntactic* negative-linear subcategory, and $\mathcal{E}[L \dashv R]_I^{\ominus}$ the *semantic* negative-linear subcategory.) The inclusions $\iota_O^{\ominus}$, $\iota_O^{\oplus}$, $\iota_{\mathcal{E}}^{\ominus}$, and $\iota_{\mathcal{E}}^{\oplus}$ extend to functors:

$$\begin{aligned} \iota_O^{\ominus} : \mathcal{N} &\rightarrow O[L \dashv R]_I^{\ominus\omega} & \iota_{\mathcal{E}}^{\ominus} : \mathcal{N} &\rightarrow \mathcal{E}[L \dashv R]_I^{\ominus} \\ \iota_O^{\oplus} : \mathcal{P} &\rightarrow O[L \dashv R]_I^{\oplus\omega} & \iota_{\mathcal{E}}^{\oplus} : \mathcal{P} &\rightarrow \mathcal{E}[L \dashv R]_I^{\oplus}. \end{aligned}$$

These are given on morphisms by

$$\begin{aligned} \iota^{\ominus} \left(n \xrightarrow{g} m \right)^b &\stackrel{\text{def}}{=} LRn \xrightarrow{\epsilon_n \circ f} m & \iota^{\oplus} \left(p \xrightarrow{f} q \right)^b &\stackrel{\text{def}}{=} Lp \xrightarrow{Lf} Lq \\ \iota^{\ominus} \left(n \xrightarrow{g} m \right)^{\sharp} &\stackrel{\text{def}}{=} Rn \xrightarrow{Rf} Rm & \iota^{\oplus} \left(p \xrightarrow{f} q \right)^{\sharp} &\stackrel{\text{def}}{=} p \xrightarrow{f \circ \eta_q} RLq. \end{aligned}$$

Proof. The linearity and thunkability conditions follow from naturality of ϵ/η (by Lemma 4). The functor laws follow from those of R/L . $\square$

Lemma 6. The inclusion functors $\iota_O^{\ominus}$ and $\iota_O^{\oplus}$ are equivalences on objects. $\square$

Lemma 7. The inclusion functors $\iota_{\mathcal{E}}^{\ominus}$ and $\iota_{\mathcal{E}}^{\oplus}$ are split essentially surjective. $\square$

Lemma 6 does not generalize to the envelope duploid directly; only certain negative envelope objects are of the form $\iota_{\mathcal{E}}^{\ominus} n$, even if $\mathcal{N}$ and $\mathcal{P}$ are univalent (by Remark 9):

Lemma 8. Let $\mathcal{N}$ and $\mathcal{P}$ be univalent. Every object $a : \mathcal{E}[L \dashv R]$ with a negative polarization choice is identified with $\uparrow a \equiv \iota^{\ominus}(\pi^{\ominus} a)$. Dually, every object $b : \mathcal{E}[L \dashv R]$ with a positive polarization choice is identified with $\downarrow b \equiv \iota^{\oplus}(\pi^{\oplus} a)$.

Proof. We show the former. We need to identify

$$\left(\pi^{\oplus} a \xrightarrow{\text{cross } a} \pi^{\ominus} a \right) = \left(R(\pi^{\ominus} a) \xrightarrow{\text{cross}(\uparrow a)} \pi^{\ominus} a \right). \quad (3)$$

By assumption, $\text{cross } a^{\sharp} : \mathcal{P}[\pi^{\oplus} a, R(\pi^{\ominus} a)]$ is an isomorphism. Thus by Theorem 7 we have $\langle \pi^{\oplus} a, \text{cross } a^{\sharp} \rangle =_{\{R(\pi^{\ominus} a)\}_{\mathcal{P}}} \langle R(\pi^{\ominus} a), \text{id}_{R(\pi^{\ominus} a)} \rangle$. Since $\text{cross } a^{\sharp} = \text{id}_{R(\pi^{\ominus} a)}$ and $\text{id}_{R(\pi^{\ominus} a)} \equiv \text{cross}(\uparrow a)^{\sharp}$, Equation 3 follows from injectivity of $\sharp$. $\square$

We now state the equalizing requirement, which solves our remaining problems.

Theorem 16. The following are equivalent:

1. The functor $\iota_O^{\ominus} : \mathcal{N} \rightarrow O[L \dashv R]_I^{\ominus\omega}$ is fully-faithful.
2. ————— is an isomorphism of categories.
3. The functor $\iota_{\mathcal{E}}^{\ominus} : \mathcal{N} \rightarrow \mathcal{E}[L \dashv R]_I^{\ominus}$ is fully-faithful.
4. ————— is a strong equivalence of categories.⁹

⁹ Conditions 4 and D are not propositions in general, so "equivalent" means "if and only if" rather than "equivalence of types" for them.

5. For all $n : N$, the fork $LRLRn \xrightarrow{LR\epsilon_n} LRn \xrightarrow{\epsilon_n} n$ is a coequalizer. That is, for all objects $n, m : N$ and morphisms $f : LRn \rightarrow m$ satisfying $LR\epsilon_n \circ f = \epsilon_{LRn} \circ f$, there is a unique morphism $f^* : n \rightarrow m$ such that $\epsilon_n \circ f^* = f$.

Proof. The implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are immediate. Their converses (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) follow from Lemmas 6 and 7. The equivalences (1) $\Leftrightarrow$ (5) and (3) $\Leftrightarrow$ (5) follow from the characterizations of linear morphisms in terms of the counit ϵ of the adjunction (Lemma 4). $\square$

Theorem 17 (dual of 16). *Dually, the following are equivalent:*

- A. The functor $\iota_O^\oplus : N \rightarrow O[L \dashv R]_t^{\oplus\omega}$ is fully-faithful.
- B. --- is an isomorphism of categories.
- C. The functor $\iota_E^\oplus : N \rightarrow E[L \dashv R]_t^\oplus$ is fully-faithful.
- D. --- is a strong equivalence of categories.⁹
- E. For all $p : P$, the fork $q \xrightarrow{\eta_p} RLq \xrightarrow{RL\eta_q} RLRLq$ is an equalizer. That is, for all objects $p, q : P$ and morphisms $g : p \rightarrow RLq$ satisfying $g \circ RL\eta_q = g \circ \eta_{RLq}$, there is a unique morphism $g^* : p \rightarrow q$ such that $g^* \circ \eta_q = g$. $\square$

Definition 26 ([16]). *An adjunction satisfying conditions 1–5 of Theorem 16 is negative equalizing, and an adjunction satisfying conditions A–E is positive equalizing. An adjunction which satisfies conditions 1–5 **and** conditions A–E is fully equalizing.*

With the equalizing requirement, we can prove univalence for both the oblique and envelope duploids. The former is simple, the latter needs the converse of Lemma 3.

Theorem 18. *If $L \dashv R$ is fully equalizing, then the oblique duploid $O[L \dashv R]$ is univalent as a split duploid if and only if N and P are univalent.*

Proof. Recall that univalence as a split duploid means that $O[L \dashv R]_t^{\oplus\omega}$ and $O[L \dashv R]_t^{\oplus\omega}$ are univalent. Using Theorem 1, univalence may be transported (in both directions) across the isomorphisms $\iota_O^\oplus : N \cong O[L \dashv R]_t^{\oplus\omega}$ and $\iota_P^\oplus : P \cong O[L \dashv R]_t^{\oplus\omega}$. $\square$

Lemma 9. *If $L \dashv R$ is negative equalizing, then for all $n : N$, the equation $LR\epsilon_n = \epsilon_{LRn}$ implies that ϵ_n is an isomorphism. Dually, if $L \dashv R$ is positive equalizing, then for all $p : P$, the equation $RL\eta_p = \eta_{RLp}$ implies that η_p is an isomorphism.* $\square$

Lemma 10. *If $L \dashv R$ is positive equalizing, then every semantically negative object in $E[L \dashv R]$ has a negative polarization choice. Dually, if $L \dashv R$ is negative equalizing, then every semantically positive object has a positive polarization choice.*

Proof. We show the negative case. Let $a : E[L \dashv R]$ be a negative object with a positive polarization choice, so that $\text{cross } a^b$ is an isomorphism. Then $\text{cross } a^\# = \varphi_{(\pi^\oplus a), (\pi^\oplus a)}(\text{cross } a^b) \equiv \eta_{\pi^\oplus a} \circ R(\text{cross } a^b)$ is the composite of two isomorphisms: $R(\text{cross } a^b)$ (by assumption) and $\eta_{\pi^\oplus a}$ (from Theorem 13 and Lemma 9). $\square$

From Lemmas 8 and 10 we obtain the necessary isomorphisms. Univalence of the envelope duploid then follows from Theorems 1 and 8.

Theorem 19. *If $L \dashv R$ is fully equalizing, and $\mathcal{N}$ and $\mathcal{P}$ are univalent, then the inclusions $\iota_{\mathcal{E}}^{\ominus}: \mathcal{N} \rightarrow \mathcal{E}[L \dashv R]_I^{\ominus}$ and $\iota_{\mathcal{E}}^{\oplus}: \mathcal{P} \rightarrow \mathcal{E}[L \dashv R]_I^{\oplus}$ are isomorphisms of categories.* $\square$

Corollary 5. *If $L \dashv R$ is fully equalizing, and $\mathcal{N}$ and $\mathcal{P}$ are univalent, then the envelope duploid $\mathcal{E}[L \dashv R]$ is univalent.* $\square$

6 Conclusions and Future Work

The contributions of this work are twofold. First, we have formalized part of Munch-Maccagnoni's work on duploids [16,15] in Rocq. The formalization covers unital magmoids and two variants of duploids: single-sorted duploids and split duploids. Second, we have made novel contributions to their study in univalent foundations. We have defined what it means to be univalent for unital magmoids, and for both variants of duploids. We studied two versions of the duploid arising from an adjunction: the original "oblique duploid" construction of Munch-Maccagnoni [16] which is naturally a split duploid, and a novel "envelope duploid" construction which is a single-sorted duploid. Finally, we have shown both of these to be univalent – in their respective senses – when arising from a fully equalizing adjunction between univalent categories.

Our formalization of Munch-Maccagnoni's work ends at showing that each duploid is equivalent to one arising from an adjunction. This result is the first part of the full "structure theorem" [16]: that the category of duploids is equivalent to the category of fully equalizing adjunctions. We note that the hom-set law of categories prevents even defining the 1-category of duploids in univalent foundations, and suggests that duploids ought to be treated as a 2-category instead. Future work could include a study of the 2-category of duploids, and a 2-categorical structure theorem for duploids.

We fall short of defining a Rezk completion for duploids: a weakly equivalent univalent duploid for any duploid $\mathcal{D}$. One way could be to use Theorem 15, but Rezk-complete the categories $\mathcal{D}_I^{\ominus}$ and $\mathcal{D}_I^{\oplus}$ before assembling $\mathcal{E}[\uparrow \dashv \downarrow]$. This procedure eluded our formalization due to complexity arising from the 2-categorical nature of adjunctions.

Lastly, there is more work to be done studying use-cases for univalence of duploids. For example, we proved directly that polarity shifts are unique in a univalent duploid, and future work could study other universal properties in univalent unital magmoids.

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